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THE CONIC SECTIONS IN THE OLD JAPANESE MATHEMATICS.

By T. HAYASHI, Lecturer of Mathematics in the Tokyo Higher Normal School, Tokyo, Japan.

In a brief Note in the MONTHLY, Vol. XII, 1905, p. 166, it is remarked that "the most surprising fact about the old Japanese mathematics is that, while the most elementary parts were regarded as common property, the more advanced results were regarded as secrets which should be communicated to a very few." In fact, at most, one of the sons of one head master and two of his most highly versed disciples might be the keepers of the secrets of advanced mathematics as the successors of the head master. This curious fact, a parallel to which can be found in the Pythagorean school, interrupted the development of Japanese mathematics. But it became gradually impossible already before the Restoration of the Imperial government from the shogoon's authorities in 1868, to keep secret among a very few the proofs of theorems and the solutions of problems which they obtained from their predecessors or which they invented themselves.

Takakazu Seki (1642—1708) was the founder of the most famous school. This school was so flourishing that the mathematics of that school is the representative of Japanese mathematics and the development of mathematics in Japan was almost entirely accomplished by the scholars of that school. Seki, a contemporary of Newton and Leibnitz, invented the method of calculation called the Enri method, which well resembles the infinitesimal calculus. The now-living head master of the Seki school, Mr. Chōrin Kawakita, turned over to the library of the Tokyo Imperial University printed books and manuscript books belonging to him and containing theorems and problems kept in secret for about two hundred years. The number of these books together with those which the library bought reaches about two thousand, so that Prof. P. Harzer says that they are a sufficient guarantee of the Japanese high esteem for mathematical knowledge. From this collection of Japanese mathematical works we may now see what were the secrets among the mathematicians in old Japan. I myself have some of the works of Japanese mathematics in my library. However, I may be allowed to say that I myself do not belong to the Seki school, or any other school of old Japanese mathematics.

In writing my work entitled "A brief history of the Japanese mathematics" in *Nieuw Archief voor Wiskunde*, 1904, pp. 296—324, and 1905, pp. 325—361, (the remaining part to be immediately published), I got most of the material from Mr. T. Endō's Dainihon-Sūgakushi or History of Mathematics in Great Japan, written in Japanese in 1896. Mr. Endō is one of the few men belonging to the old school to be found now-a-days, and his work is "written in a language not entirely intelligible and sometimes even repulsive to a student of modern mathematics, and in the characteristic tone peculiar to the men of the old school and altogether at variance with the spirit of modern mathematics," as one of my masters, Prof. Dr. R. Fujisawa of the Tokyo Imperial University, says in his paper, "Note on the mathematics of the old Japanese school" in the second international congress of mathematicians held at Paris in 1900 (see E. Duporcq's *compte rendu du congres*, pp. 379—393). So I compiled in my work the materials got from Mr. Endō's work in a manner easier to comprehend and quite different from that of Mr. Endō's work.

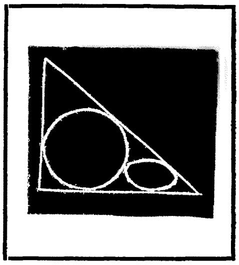
'I will take this opportunity to publish the content of the most famous work on the conic sections written about seventy years ago by a mathematician and containing curious and difficult problems of a quite different kind from those in European mathematics. But let me first give a history of the treatment of the conic sections in Japanese mathematics.

The mathematicians did not treat the sections of a right cone, as Apollonius did, but they treated the sections of a right cylinder, and therefore they knew only the ellipse until European mathematics was imported, and did not know the two curves, hyperbola and parabola, though they treated many other curves, algebraic or transcendental.

I do not know who first treated the ellipse. But we may say that Naomaru Ajima was the first mathematician who could completely calculate the length of an elliptic arc and the area of an elliptic segment, and consequently the whole perimeter and area of an ellipse, by a peculiar method which very well resembles the method of integration called by the mathematicians *Enri Method*. The literal meaning of the word Enri is "principle of a circle." Of all the problems that were investigated by the old Japanese mathematicians, the rectification and quadrature of a circle were the most attractive ones to them; so that several methods pertaining to the rectification, quadrature, and cubature of any curved line and any curved surface were brought together under the name of the Enri method, the ellipse being called *Sokuen*, considered as one kind of En or circles.

Naomaru Ajima quite reformed the hitherto prevailing method for the rectification and quadrature of a circle, the new method much more resembling the infinitesimal calculus. He was born in 1739 and died in 1798. He became the head-master of the Seki school. It was one of his most remarkable achievements that the whole perimeter of an ellipse and the length of an elliptic arc were each expressed in an infinite series.

Ajima had written many manuscripts, but there was no printed one. His best disciple Makoto Kusaka (1764-1839) compiled in 1799 the manuscripts into a book which he named Fukyū-Sampō, which means a perpetual mathematics. This however, was not published as intended. The later scholars never failed to copy it and regarded it as one of the most esteemable works. It contains six problems relating to ellipses. The following enunciations will have something of interest, as indicating what problems the old mathematician solved. [The mathematicians of the old school did not enunicate problems minutely, but they drew always such a figure for every problem that we can understand the meaning of the problem; so they usually put the words "as in the figure" in the first part of the enunciation.]



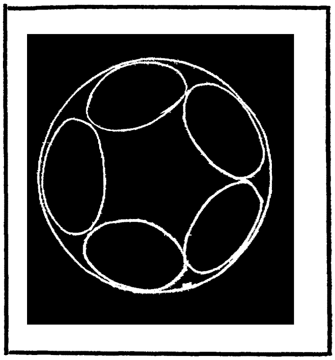
The 6th problem. As in the figure, a circle and an ellipse are inscribed in a right-angled triangle. Given the two sides of the triangle and the minor axes, find the major axes.

The 11th problem. An ellipse and a circle are inscribed in a rectangle, the major axis of the ellipse lying along the diagonal of the rectangle. Given the two sides of the rectangle and the minor axis, find the diameter of the circle.

The 20th problem. A spheroid is cut by a plane. Given the axes of the spheroid and the axes of the section, find the volume of one part of the spheroid.

The 21st problem. A spheroid is inscribed in a rectangular parallelopipedon. Given the three sides of the parallelopipedon and the minor axes of the spheroid, find the major axis.

The 22nd problem. In a given ellipse two equal chords and two equal circles are drawn tangent to the chords and the ellipse. Given the major and minor axes of the ellipse and diameter of the circles, find the length of the chords.



The 23rd problem. As in the figure, a certain number of equal ellipses are inscribed in a circle. Given the diameter of the circle and the major axis and the number of the ellipses, find the minor axis.

Kōhan Sakabe was another of the best pupils of Ajima. He died in 1824, the date of his birth being unknown. In 1810, Sakabe wrote Sampō-Tenzan-Shinan (a guide to the Tenzan method, which resembles the algebra.) One hundred and ninety-six typical problems are arranged in order from easy problems to difficult ones. At the end of the work, we find the method of finding the length of the whole perimeter and of an arc of an ellipse which had been already discovered by his master Ajima. This was the first appearance of the problems pertaining to ellipses in printed books. It is said by some scholars that the part of the book relating to ellipses had been written by his pupil, Hisanori Kawai.

The successor of Kusaka was Yasushi Wada (or Nei Wada) who died in 1840, the date of birth being unknown. The Enri method was quite reformed by him. His improvement of the method was epoch-making in the history of Japanese mathematics, as Ajima's was. I will next explain his method of finding the perimeter of an ellipse, which is contained in his Enri-shinkō or Elementary lessons on the Enri method.

Divide the major axis $AB=a$ into $2n$ equal parts. Let PQ and $P'Q'$ be the mth and $m+1$th ordinates (called chō), the numbers being given to the ordinates from the minor axis $=b$, taken as the first ordinate to the right-hand ones and to the left-hand ones.

The $(m+1)$th Chōkaku, that is

$$m\text{th chō} - (m+1)\text{th chō} = \left(\frac{m}{n}\right) b \div \sqrt{1-\left(\frac{m}{n}\right)^2}.$$

The $(m+1)$th Haikaku, that is $2\times$ elliptic arc intercepted between the mth and $(m+1)$th chō equals

$$\frac{a}{n}\sqrt{1-\left(\frac{m}{n}\right)^2\left(1-\frac{b^2}{a^2}\right)} \div \sqrt{1-\left(\frac{m}{n}\right)^2}.$$

By the binomial expansion,

$$\left[1-\left(\frac{m}{n}\right)^2\left(1-\frac{b^2}{a^2}\right)\right]^{\frac{1}{2}} = 1-\tfrac{1}{2}\left(\frac{m}{n}\right)^2\left(1-\frac{b^2}{a^2}\right)-\frac{1.1}{2.4}\left(\frac{m}{n}\right)^4\left(1-\frac{b^2}{a^2}\right)^2$$

$$-\frac{1.1.3}{2.4.6}\left(\frac{m}{n}\right)^6\left(1-\frac{b^2}{a^2}\right)^4-\frac{1.1.3.5}{2.4.6.8}\left(\frac{m}{n}\right)^8\left(1-\frac{b^2}{a^2}\right)^6-\dots$$

and

$$\left[1-\left(\frac{m}{n}\right)^2\right]^{-\frac{1}{2}} = 1+\tfrac{1}{2}\left(\frac{m}{n}\right)^2+\frac{1.3}{2.4}\left(\frac{m}{n}\right)^4+\frac{1.3.5}{2.4.6}\left(\frac{m}{n}\right)^6+\dots$$

By actual multiplication, $\left[1-\left(\frac{m}{n}\right)^2\left(1-\frac{b^2}{a^2}\right)\right]^{\frac{1}{2}}\times\left[1-\left(\frac{m}{n}\right)^2\right]^{-\frac{1}{2}}$ is expanded into a power series of $(m/n)^2$. Then $\frac{1}{2}\times$perimeter of the ellipse$=\lim_{n=\infty}\sum^{m}$ (mth

$$\text{haikaku})=\pi a\left[1-\frac{1}{2}\cdot\frac{1}{2}\left(1-\frac{b^2}{a^2}\right)-\frac{1.1}{2.4}\cdot\frac{1.3}{2.4}\left(1-\frac{b^2}{a^2}\right)^2-\frac{1.1.3}{2.4.6}\cdot\frac{1.3.5}{2.4.6}\left(1-\frac{b^2}{a^2}\right)^3\cdots\right]$$

This was rewritten as follows:

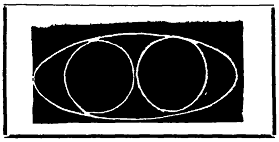
$$\tfrac{1}{2}\times\text{perimeter}=\pi a\left\{1-\frac{1.1}{1^2}\cdot\frac{1-(b^2/a^2)}{4}-\frac{1.1}{1^2}\cdot\frac{1.3}{2^2}\left[\frac{1-(b^2/a^2)}{4}\right]^2-\frac{1.1}{1^2}\cdot\frac{1.3}{2^2}\cdot\frac{3.5}{3^2}\right.$$
$$\left.\left[\frac{1-(b^2/a^2)}{4}\right]^3-\frac{1.1}{1^2}\cdot\frac{1.3}{2^2}\cdot\frac{3.5}{3^2}\cdot\frac{5.7}{4^2}\left[\frac{1-(b^2/a^2)}{4}\right]^4-\cdots\right\}$$

In this series there appears the factor $1-(b^2/a^2)$, that is the square of eccentricity. This factor was called by the mathematician *ritsu* or modulus of the series.

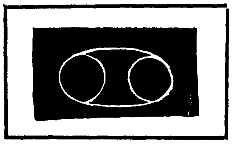
Kyō Uchida, well known by the name of Gokwan Uchida and living up to about 1877, was one of the best pupils of Kusaka and Wada. Kubota, a disciple of Uchida, wrote the first part of Enri-Shōhei-jutsu (the date of publication is not known). In 1855, the second part of the same work was written by Kumamoto, his fellow-student, and revised by Takemura of the same school. The first part is devoted to the methods of finding the center of gravity of plane figures and the second part to those of solid figures. Among the latter, the following problem is found: A spheroid is cut off by a plane parallel to one of its principal planes. Find the center of gravity of the remaining part.

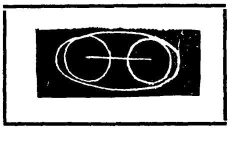
Hiroshi (or Kwan) Hasegawa (1782—1832) was also one of the best pupils of Kusaka, but was expelled from Kusaka's school on account of his bad conduct, and was scorned by the mathematicians of the Seki school. Nevertheless he continued his study. He published many books by the names of his pupils. By the name of Tsunemitsu Murata, he published Sampō-Sokuen-Shōkai in 1831. Fifty problems relating to ellipses are arranged. This work was very famous and was respected by contemporary and later mathematicians. It is the most elegant work devoted to the study of ellipses. It will be interesting for us to know the content of this work, in order to learn the characteristic inclination of study of mathematics peculiar to the old Japanese school.

(1.) As in the figure, two equal circles in external contact are inscribed in an ellipse. Given the major and minor axes, fine the radius of the circles.



(2.) As in the figure, the circles in contact with an ellipse at the extremities of the major axis and having the maximum radius are drawn. Given the major and minor axes of the ellipse, find the maximum radius. [The circles are the osculating circles].

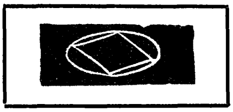




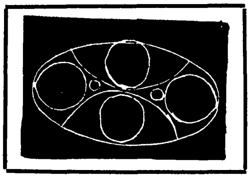
(3.) As in the figure, two equal circles not in contact are inscribed in an ellipse. Given the major and minor axes and the radius of the circles, find the distance of the centers of the two circles.

(4.) Two unequal circles in external contact are inscribed in an ellipse. Given the major and minor axes and the radius of one circle, find the radius of the other circle.

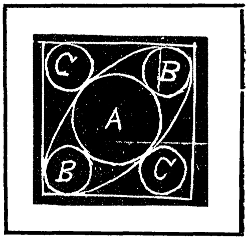
(5.) A rhombus is inscribed in an ellipse. Given the major and minor axes and the area of the rhombus, find the side of the rhombus.



(6.) As in the figure, two equal circular arcs are drawn and four equal large circles and two equal small circles are inscribed within an ellipse. Given the radius of the small circles, find the maximum of the radius of the large circles [two of the large circles are the osculating circles at the extremities of the major axis.]



(7.) As in the figure, an ellipse, one circle A, two equal circles B and two other equal circles C are drawn within a square. Given the radius of the circle C, find the maximum radius of the circle B [the circles B are the osculating circles at the extremities of the major axis.]



(8.) Four equal ellipses and five equal circles are drawn within a square, the extremities of the major axes of two neighboring ellipses being the same point. One of the equal circles has its center at the center of the square to which circle the four ellipses are tangent externally, the other four circles are in the areas formed by the intersection of pairs of ellipses. Given the major axis, find the minor axis.

(9.) Three equal ellipses and one circle are drawn within an equilateral triangle. The major axes of the ellipse coincide with the median of the triangle and the circle is tangent internally to the three ellipses. Given the major and minor axes of the ellipses, find the diameter of the circle.

(10.) As in the figure (23rd problem, p. 173), a certain number *e.g.* five, of equal ellipses are drawn within a circle. Given the diameter of the circle and the number of ellipses, find the major and minor axes.

(11.) As in the figure (23rd problem, p. 173), a certain number *e.g.* five, of equal ellipses are drawn within a circle). Given the major and minor axes, find the number of the ellipses when the diameter of the external circle is minimum.

(12.) Four equal circles are inscribed within the external curvelinear triangular spaces formed by two intersecting equal ellipses [having the same centre and the major axes at right angles]. Given the major axis, find the minor axis when the diameter of the circles is maximum.

(13.) An ellispe and a square are inscribed in a half isosceles trapezoid [the half part of an isosceles trapezoid divided by the join of the middle points of the two parallel sides]. Given the two parallel sides and the distance of the two parallel sides, find the side of the square.

(14.) Two equal chords not intersecting within the ellipse and two unequal circles are drawn within an ellipse, the circles being in internal contact with the ellipse at the extremities of the minor axis and tangent to the chords. Given the major and minor axes and the diameters of both circles, find the length of the chords.

(15.) The enunciation is the same as the preceeding; except that the chords intersect within an ellipse.

(16.) Two equal intersecting chords, two equal ellipses and two unequal circles are drawn within a circle, the circles and ellipses being tangent to the given circle and the intersecting chords. Given the diameter of the external circle and the major and minor axes, find the diameters of the two unequal circles.

(17.) Two circles *A* and *B* are intersecting, the center of *B* lying on the circumference of *A*; and two equal chords, two equal ellipses tangent to *A* and the two intersecting chords, and two equal circles *C* tangent to *A*, and one small circle *D* tangent to *C* and *B* and the intersecting chords are drawn, the major axes being parallel to the chords respectively. Given the major and minor axes, and the diameter of the circle *A*, find the diameter of the circle *D*.

(18.) An ellipse is inscribed in a circle and then a circle *B* is described on the minor axis of the ellipse. Two equal non-intersecting chords are drawn tangent to *B* and two equal circles *D* are inscribed in the smaller segments of the ellipse, cut off by the equal chords. Circles *A* and *C* are drawn tangent to the equal chords tangent externally to the ellipse and tangent internally to the given circle. Given the diameters of the given circle and the circle *B*, find the diameter of the circles *D*.

(19.) Two equal lines are drawn through the center of a rectangle. In the two pentagonal spaces two equal ellipses are drawn the major axis of which is equal and parallel to the shorter side of the rectangle. In the two triangular spaces, two equal circles *A* are drawn and two equal circles *B* are drawn tangent to the ellipses and the two intersecting lines. Given the two sides of the rectangle and the diameter of the circle *A*, find the diameter of the circle *B*.

(20.) Two equal ellipses are inscribed in a circle in such a way that three equal pentagons can be inscribed in the segments of the ellipse, a side of each pentagon lying in a common line and one pentagon being common to the two ellipses. Given the diameter of the circle, find the side of the pentagon and the major and minor axes of the ellipses.

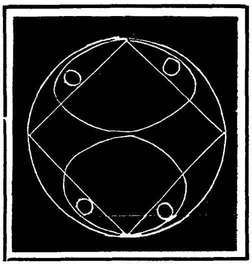
(21.) Two equal pentagons are inscribed in an ellipse in such a way that each of two diameters of the ellipse forms a side each of the two pentagons. Four equal circles are inscribed in the segments cut off by the sides of the pentagons. Given the major axis of the ellipse, find the diameter of the circles.

(22.) Six equal chords, two of which are parallel to the major axis, are drawn within an ellipse, and four equal circles are drawn in the four equal segments. Given the major and minor axes, find the diameter of the circles.

(23.) A chord is drawn in an ellipse and three unequal circles *A*, *B*, and

C are drawn, *A* and *B* each touching the ellipse in two points and tangent to the chord and *C* touching the ellipse in one point and tangent to the chord. Given the minor axis and the diameter of the circles *A* and *B*, find the diameter of the circle C.

(24.) Five unequal circles are drawn within an ellipse the circles touching the ellipse in two points, and the four chords *A*, *B*, *C*, *D* touching the two neighboring circles are drawn. Given the three chords *A*, *B*, *C*, find the fourth chord *D*.



(25.) As in the figure, a square, two equal ellipses and four equal circles are drawn within a circle. Given the diameter of the external circle, find the maximum diameter of the equal circles.

(26.) Two equal circular arcs and four equal small circles are inscribed in the segments of the ellipse cut off by these circular arcs are drawn within an ellipse. Given the major and minor axes, find the diameter of the small circles.

(27.) Two intersecting lines from the vertices of the acute angles and two equal ellipses having the major axes parallel to one side are drawn within a right-angled taiangle. Given the ratio of the major axis to the one side parallel to it, find the ratio of the minor axis to the other side parallel to it.

(28.) Two equal circles *A* with centers on the major axis, two equal circles *B* with centers on the minor axis, and one circle *C* whose center is the center of ellipse and tangent to circles *A* and *B*, are drawn within an ellipse. Given the major and minor axes, find the diameter of the circles *A*.

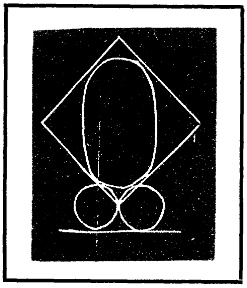
(29.) Circles whose centers are on major axis, and squares one of whose diagonals coincide with major axis are inscribed alternately within an ellipse. Given the major and minor axes, find the number of circles and the number of squares.

(30) and (31.) Two subsidiary problems of the preceeding.

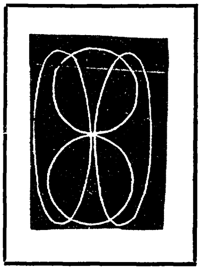
(32.) An ellipse and a circle *A* are inscribed in a semi-circle, the major axis of the ellipse being parallel to the base of the semi-circle and the diameter of circle *A* coinciding with the minor axis of the ellipse. Two equal circles *B* tangent externally are inscribed in each of the crescents formed by the ellipse and circle *A*. Given the diameter of the small circles *B*, find the diameter of the semi-circle when the major axis is a maximum.

(33.) An ellipse is drawn within a square in such a way that it touches three sides of the square and its major axis is parallel to the side of the square. Four equal circles *B* whose centers form a square whose sides are parallel to the former square are inscribed in an ellipse, also a fifth equal circle *B* is discribed on the minor axis produced so that it is tangent to the ellipse and the side of the square. Two equal circles *A* are drawn tangent to this fifth circle *B* and two sides of the square. Given one side of the square, find the diameter of the circles *A*.

(34.) As in the figure, two equal circles, an ellipse and a square are drawn above a straight line, the ellipse touching the four sides of the the square and two circles. Given the diameter of the circles and the minor axis, find the major axis.

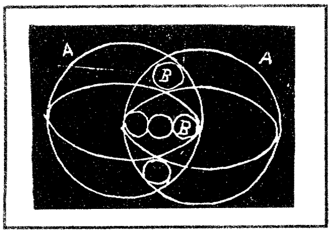


(35.) As in the figure, two equal ellipses and two equal circles are drawn. Given the major and minor axes, find the diameter of the circles.



(36.) As in the figure, two equal circles *A* and two equal ellipses and five equal small circles *B* are drawn. Given the diameter of the circles *A*, find the limiting length of the diameter of the small circles *B*.

(37.) As in the figure, two equal circles and one ellipse are drawn above a straight line, the major axis being parallel to the straight line. Given the diameter of the circles and the minor axis, find the major axis.



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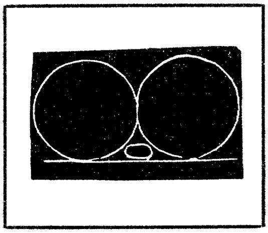

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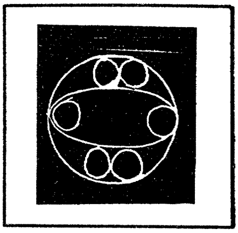

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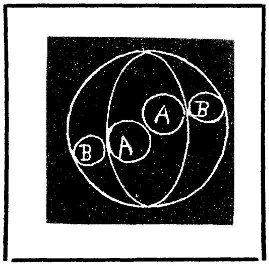
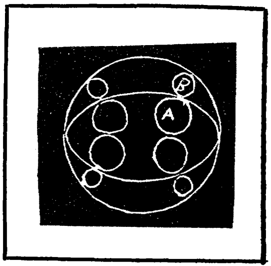
(38.) As in the figure, an ellipse and six equal circles are drawn within a circle. Given the diameter of the external circle, find the maximum diameter of the equal circles.

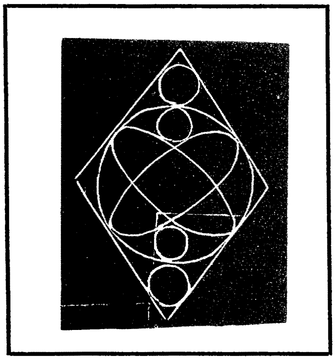


(39.) As in the figure, an ellipse, two equal circles *A* and two equal circles *B* are drawn within a circle. Given the diameters of the circles *A* and *B*, and the major axis, find the minor axis.



(40). As in the figure, an ellipse, four equal circles *A* and four equal circles *B* are drawn within a circle. Given the diameters of the circles *A* and *B*, and the major axis, find the minor axis.

(41.) As in the figure, a circle, two equal ellipses, two equal circles *A* and two equal circles *B* are drawn within a rhombus. Given the major and minor



axes, and the diameter of the circle A, find the diameter of the circles B.

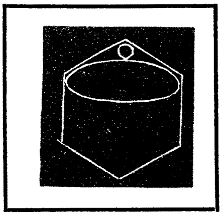
(42.) A spheroid and a certain number of equal small spheres surrounding the spheroid are drawn within a sphere. Given the major and minor axes, and the diameter of the small spheres, find the number of the small spheres.

(43.) Equal ellipses whose major axes coincide with diameters joining opposite vertices of a regular incribed polygon are inscribed within a circle. Equal circles are inscribed in the triangular space formed by two consecutive ellipses and the circle. Given the major and minor axes of the ellipses and the diameter of the small circles, to find the number of ellipses.

(44.) Equal ellipses whose major axes coincide with the diameters joining opposite vertices of a regular inscribed polygon are inscribed in the polygon. In the triangular spaces, formed at the vertices of each ellipse, by three consecutive ellipses, equal circles B are drawn, and in the space common to all the ellipses at the center of the polygon a circle A is drawn. Given the diameter of the external circle, find the diameter of the circle A when the diameter of the circles B is a maximum.

(45.) Two equal spheriods whose major axes are at right angles, and a certain number of equal spheres touching each of the spheroids and the sphere are drawn within a given sphere, the small spheres touching each other in pairs. Given the diameter of the external sphere and the number of the small spheres, find the diameter of the small spheres.

(46.) A circle A is concentric with a given circle. A certain number of equal ellipses tangent to the two circles at the extremities of the major axis are drawn. In the space common to two consecutive ellipses equal circles B are drawn. Given the diameter of the external circle, the major and minor axes, and the number of the ellipses, find the diameter of the circles B.



(47.) As in the figure, an ellipse and a circle are inscribed within a regular polygon. Given the major and minor axes, find the diameter of the circle.

(48.) Equal ellipses are drawn in a regular polygon. The number of ellipses is equal to the number of sides of the polygon; each ellipse is tangent to three consecutive sides of the polygon; and the major axis of each ellipse is parallel to the mid side of the three consecutive sides. In the areas common to two consective ellipses equal circles B are drawn. At the center of the polygon a circle A is drawn tangent to the ellipses. Given the side of the polygon, and the diameter of the circle A, find the diameter of the small circles.

(49.) Ellipses are incribed in a polygon as in problem 47. Given the major and minor axes, find the side of the polygon.

(50.) Equal ellipses are drawn within a polygon as in problem 49, and equal circles are drawn in the area common to two consecutive ellipses. Given the major and minor axes, find the diameter of the circles.*

As an appendix of this book, the author Murata described the minute demonstrations of the solutions of the sixth and eleventh problems of Ajima's Fukyū-Sampō in which the solutions of all problems were not demonstrated.

I will conclude this paper by adding that ellipsoids which were treated by Japanese mathematicians of the old school were spheroids only, called Chōryūen or elongated solid circle.

Tokyo, *December 1, 1905.*

ON THE FUNCTIONS WHICH HAVE A GIVEN ALGEBRAICAL ADDITION THEOREM.

By M. KABÀ, Tokyo, Japan.

Let $f(x)$ denote the uniform analytical function possessing a given algebraical addition theorem, that is,

$$f(x+y)=F[f(x),f(y)]\ldots(1).$$

It is evident that the function satisfies the equation

$$f(2x)=F[f(x),f(x)]\ldots(2).$$

In the first place we prove conversely that the uniform analytical function $\varphi(x)$ which satisfies equation (2) will satisfy equation (1). Set

$$\varphi(x+y)-F[\varphi(x),\ \varphi(y)]=F_1(x,\ y).$$

If we use $x+h$ instead of y and develop the function F_1 in a power series of h by Maclaurin's theorem, all the coefficients will be zero. In fact, as the function F is symmetrical in $\varphi(x)$ and $\varphi(y)$, we have the following:

$$\left[\frac{dF}{dh}\right]_{h=0}=\frac{1}{2}\frac{d}{dx}F[\varphi(x),\ \varphi(x)],\quad \left[\frac{d^2F}{dh^2}\right]_{h=0}=\frac{1}{2^2}\frac{d^2}{dx^2}F[\varphi(x),\ \varphi(x)],\ \ldots$$

And also we have the following:

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*In justice to the author and to Editor Dickson, I wish to state that I changed the enunciation of all the problems unaccompanied by diagrams, to avoid making so many wood-cuts. Ed. F.